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Asymptotic analysis of an elastic rod with rounded ends

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We derive a one-dimensional model for an elastic shuttle, that is, a thin rod with rounded ends and small fixed terminals, by means of an asymptotic procedure of dimension reduction. In the model, deformation of the shuttle is described by a system of ordinary differential equations with variable degenerating coefficients, and the number of the required boundary conditions at the end points of the one-dimensional image of the rod depends on the roundness exponent $m \in (0, 1)$. Error estimates are obtained in the case $m \in (0, 1/4)$ by using an anisotropic weighted Korn inequality, which was derived in an earlier paper by the authors. We also briefly discuss boundary layer effects, which can be neglected in the case $m \in (0, 1/4)$ but play a crucial role in the formulation of the limit problem for $m \geq 1/4$.

KEYWORDS

elliptic equations and systems, Korn inequality, linear elasticity system, mechanics of deformable solids, roundness exponent, thin rod

MSC CLASSIFICATION

74B05

1 | INTRODUCTION

The elasticity theory of thin rods is an important applied discipline having a long history starting from the work of Klebsch,¹ which laid the foundations of the modern theory of elastic rods. There still remain open questions on the validity and accuracy of approximation used in the technical theories and engineering calculations for thin elastic objects. The mathematical model of a thin elastic rod with an explicit small geometric parameter is interesting and nontrivial, and it can be systematically and rigorously treated using asymptotic analysis based on a system of partial differential equations. This is also the approach of the present work. The scheme includes dimension reduction of the three-dimensional elasticity system into a one-dimensional problem, compare Bermúdez and Viaño,² Tutek and Aganovich,³ Trabuco de Campos and Viaño,⁴ Le Dret,⁵ Sanchez-Hubert and Sanchez-Palencia,⁶ Nazarov and Slutskiy,⁷ Nazarov,⁸ Panasenko⁹ and many other publications, and it is straightforward at the first glance; however, in the one-dimensional model, the roundness of the rod ends makes the coefficients of the differential operators to degenerate, which requires serious modifications in the solvability and justification arguments.

We consider a thin elastic rod having ends of the paraboloidal shape, and our aim is an asymptotic analysis of this object. In particular, we will observe the peculiar phenomenon that the shape of the ends described by its roundness exponent $m \in (0, 1)$ (see Section 2.1, Formula 2) leads to losses of some boundary conditions in the one-dimensional model. This phenomenon is known to occur in the case of the linear elasticity system for plates with sharp edges (see Makhov,¹⁰ Mikhlin,¹¹ and Campbell et al.¹²) but was not investigated yet for rods with rounded ends; notice that $m = 0$ and $m = 1$ correspond to straight and conical ends, respectively.

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A Korn inequality for a thin elastic rod with rounded ends was derived in Nazarov et al.¹³ The inequality was formulated using a special weighted anisotropic norm depending on m , and three critical values, $m = 1/4$, $m = 1/2$, and $m = 3/4$, were found, such that the anisotropic norms change crucially at these values of m . The asymptotic analysis proposed in the present paper leads to a one-dimensional system with the same critical values of m . The number of boundary conditions at each end varies from 6 (for $0 < m < 1/4$) to 0 (for $3/4 \leq m < 1$). A more detailed analysis of the behavior of the solution near the end would need a study of the boundary layer phenomenon, which is only briefly considered in Section 3.4; we also refer to Nazarov,¹⁴ where the same critical values of the exponent m were detected in description of behavior of elastic fields in paraboloids at infinity.

In Section 2, we perform the formal asymptotic analysis of the deformations of a thin elastic anisotropic rod. The dimension reduction procedure provides a system of four ordinary differential equations, where the coefficients are degenerate due to the roundness of the rod ends. The calculations are presented explicitly in the case of an isotropic rod. In Section 3, a careful inspection of the equations shows that, first, the number of necessary boundary conditions depends on the roundness exponent m and, second, the one-dimensional limit problem is uniquely solvable only in the case $m \in (0, 1/4)$ but does not have this important property for $m \in [1/4, 1)$ and thus requires a modification. In this paper, we are able to complete the analysis in the case $m \in (0, 1/4)$ by proving asymptotically sharp error estimates based on an anisotropic weighted Korn inequality from.¹³ The case $m \geq 1/4$ is postponed to a planned forthcoming paper by the authors, since it seems to require a careful investigation of the boundary layer phenomenon and the techniques of self-adjoint extensions of differential operators for the unique solvability of the one-dimensional limit problem, compare Nazarov.¹⁵ The procedure to derive the four ordinary differential equations (43) is well known; see Nazarov and Slutski⁷ and Nazarov⁸, Ch. 5, & 3. We prefer here a self-contained presentation and thus repeat the details in order to demonstrate the reason for the degeneration of the coefficients.

It should be mentioned that boundary value problems with degenerating coefficients, leading to the loss of boundary conditions, have been studied already for a long time; see, for example, Keldyž,¹⁶ Oleinik,¹⁷ Mihlin,¹⁸ Vižik,^{19,20} Smirnov,²¹ Kohn,²² Vižik and Gružin,²³ and many others. However, these references deal with higher dimensional boundary value problems in contrast to our case, where the limit problem is an ordinary differential equation, which can be treated by quite elementary tools.

The stationary problem studied in the present paper describes the deformations of a thin rod with rounded ends under a volume force. As further motivation of our paper, we remark that it forms a preliminary step in the study the corresponding spectral elasticity problem, which leads to a rather interesting question: as is known by Nazarov,²⁴ Bakharev and Nazarov,²⁵ Cardone et al.,²⁶ the spectrum of a cuspidal elastic body may become continuous, causing the so called Vibrational Black Holes (see Nazarov,²⁴ Mironov,²⁷ Krylov,²⁸ and others). Indeed, the open question whether the one-dimensional model inherits the properties of the band-gap spectrum in the three-dimensional elasticity problem surely deserves to be investigated.

Finally, in the cases of thin elastic rods with conical ($m = 1$) or peak-shaped ($m > 1$) ends, we expect that there appear phenomena which are different from the present case and which cannot be treated by any simple modifications of the approach of the present paper; thus, these cases will not be discussed here.

2 | FORMAL ASYMPTOTIC ANALYSIS

2.1 | Formulation of the problem

Let ω be a domain in the plane \mathbb{R}^2 bounded by a simple smooth closed contour $\partial\omega$, and let H be a smooth function (of class $C^\infty[-1, 1]$ for simplicity) such that $H(z) > 0$ for $z := x_3 \in (-1, 1)$, while $H(\pm 1) = 0$. We define a rounded rod, the shuttle,

$$\Omega_h = \{x = (x_1, x_2, x_3) = (y, z) \in \mathbb{R}^3 : |z| < 1, h^{-1}H(z)^{-1}y \in \omega\}, \quad (1)$$

where $h \in (0, 1]$ is a small parameter and $y = (y_1, y_2) = (x_1, x_2)$ are the Cartesian coordinates in the plane normal to the axis of the rod. Note that x_j and h are dimensionless because we have rescaled the length of the rod to 2. We assume that the rod Ω_h has curved ends, namely, for some roundness parameter $m \in (0, 1)$,

$$H(z) = (1 \mp z)^m \mathbf{H}_\pm(\pm z) \quad \forall \pm z \in [0, 1], \quad \text{where } \mathbf{H}_\pm \in C^\infty[0, 1], \quad H_\pm := \mathbf{H}_\pm(\pm 1) > 0. \quad (2)$$

In the formulation of the elasticity problem, we systematically use the Voigt–Mandel (matrix) notation^{8,29,30} instead of the tensor notation, since that is much more convenient for our asymptotic constructions. Treating the displacement vector u in the fixed Cartesian system x as a column vector $(u_1, u_2, u_3)^T$, where T stands for the transpose, we consider the elastic strain and stress columns of height 6

$$\varepsilon = (\varepsilon_{11}, \varepsilon_{22}, \sqrt{2}\varepsilon_{12}, \sqrt{2}\varepsilon_{13}, \sqrt{2}\varepsilon_{23}, \varepsilon_{33})^T, \quad \sigma = (\sigma_{11}, \sigma_{22}, \sqrt{2}\sigma_{12}, \sqrt{2}\sigma_{13}, \sqrt{2}\sigma_{23}, \sigma_{33})^T, \quad (3)$$

where

$$\varepsilon_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right), \quad i, j = 1, 2, 3, \quad (4)$$

are the Cartesian components of the strain tensor (ε_{ij}) . The factors $\sqrt{2}$ are inserted into (3) in order to equalize the natural norms of tensors and columns. According to (3) and (4), we have

$$\mathcal{Z}(u) = D(\nabla_x)u,$$

where $\nabla_x = (\partial_1, \partial_2, \partial_3)^T$, $\partial_j = \frac{\partial}{\partial x_j}$, and $D(\nabla_x)$ is a 6×3 matrix of first-order differential operators, and

$$D(\xi_1, \xi_2, \xi_3)^T = \begin{pmatrix} \xi_1 & 0 & 2^{-1/2}\xi_2 & 2^{-1/2}\xi_3 & 0 & 0 \\ 0 & \xi_2 & 2^{-1/2}\xi_1 & 0 & 2^{-1/2}\xi_3 & 0 \\ 0 & 0 & 0 & 2^{-1/2}\xi_1 & 2^{-1/2}\partial_2 & \xi_3 \end{pmatrix}. \quad (5)$$

Hooke's law takes the form $\sigma(u) = A\varepsilon(u)$ where A is the stiffness matrix composed of the elastic material moduli. This matrix of size 6×6 is symmetric and positive definite. In particular, the Lamé constants $\lambda \geq 0$ and $\mu > 0$ of a homogeneous and isotropic material are contained in this matrix as follows:

$$A = \begin{pmatrix} \lambda + 2\mu & \lambda & 0 & 0 & 0 & \lambda \\ \lambda & \lambda + 2\mu & 0 & 0 & 0 & \lambda \\ 0 & 0 & 2\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & 2\mu & 0 & 0 \\ 0 & 0 & 0 & 0 & 2\mu & 0 \\ \lambda & \lambda & 0 & 0 & 0 & \lambda + 2\mu \end{pmatrix}. \quad (6)$$

The elasticity equations read as follows:

$$L(\nabla_x)u(h, x) := D(-\nabla_x)^T A D(\nabla_x)u(h, x) = f(h, x), \quad x \in \Omega_h, \quad (7)$$

where $f = (f_1, f_2, f_3)^T$ is a vector column of volume forces. Let us describe the boundary conditions. We denote the surface of the rod by $\partial\Omega_h$ and suppose that the rod Ω_h is clamped over the non-empty surfaces $\Gamma_h^\pm \subset \mathcal{B}_{Cr(h)}^\pm$, the terminals,

$$u(h, x) = 0, \quad x \in \Gamma_h^\pm, \quad (8)$$

where $\mathcal{B}_R^\pm = \{x : |y|^2 + |z \mp 1|^2 < R^2\}$ and $\Gamma_h^\pm \supset \partial\Omega_h \cap \mathcal{B}_{r(h)}^\pm$ with $r(h)$ specified in Section 3.4. On the lateral surface $\Gamma_h = \partial\Omega_h \setminus (\Gamma_h^+ \cup \Gamma_h^-)$, we impose the traction-free boundary conditions

$$B(h, x, \nabla_x) := D(n)^T A D(\nabla_x)u(h, x) = 0, \quad x \in \Gamma_h, \quad (9)$$

where $n = (n_1, n_2, n_3)^T$ is the unit outward normal vector.

2.2 | Decomposition of differential operators and asymptotic ansatz

The change of coordinates $(x_1, x_2, x_3) \mapsto (\eta_1, \eta_2, z) = (h^{-1}y_1, h^{-1}y_2, z)$ leads to the splitting of the operator $D(\nabla_x)$, namely,

$$D(\nabla_x) = h^{-1}D_\eta + D_z, \quad \text{where } D_\eta = D(\nabla_\eta, 0), \quad D_z = D(0, 0, \partial_z), \quad (10)$$

where D is as in (5). Thus, in view of (10), the matrix differential operator of size 3×3 on the left-hand side of (7) is represented as follows:

$$\begin{aligned} L(x, \nabla_x) &= h^{-2}L^0(\eta, z, \nabla_\eta) + h^{-1}L^1(\eta, z, \nabla_\eta, \partial_z) + h^0L^2(\eta, z, \partial_z), \\ L^0 &= -D_\eta^\top AD_\eta, \quad L^1 = -D_\eta^\top AD_z - D_z^\top AD_\eta, \quad L^2 = -D_z^\top AD_z. \end{aligned} \quad (11)$$

Let $\nu = (\nu_1, \nu_2)^\top$ be the outward unit normal vector of the boundary $\partial\omega$ of $\omega \subset \mathbb{R}^2$. Then the normal vector n on the surface Γ_h is of the form

$$N(h, x)^{1/2}n(h, x) = (\nu_1, \nu_2, h\partial_z H(z))^\top, \quad \text{where} \quad N(h, x) = 1 + h^2\partial_z H(z)^2.$$

Hence,

$$N^{1/2}D(n) = D(\nu_1, \nu_2, 0) + hD(0, 0, \partial_z H(z)) =: D_\nu + hD_0.$$

According to the last formula and the splitting (10), we have

$$\begin{aligned} N(h, x)^{1/2}B(h, x, \nabla_x) &= h^{-1}B^0(\eta, z, \nabla_\eta) + h^0B^1(\eta, z, \nabla_\eta, \partial_z) + h^1B^2(\eta, z, \partial_z), \\ B^0 &= D_\nu AD_\eta^\top, \quad B^1 = D_\nu AD_z^\top + D_0 AD_\eta^\top, \quad B^2 = D_0 AD_z^\top. \end{aligned} \quad (12)$$

Throughout the paper, we make use of the “key” identity inherited from (5)

$$D_z^\top e_{(i)} = D_\eta^\top e_{(3)} \eta_i \partial_z, \quad i = 1, 2, \quad (13)$$

where $e_{(j)}$ is the j th canonical unit vector. We assume that

$$f(h, x) = h^{-1}f^0(\eta, z) + h^0\bar{f}^0(z) + \tilde{f}(h, x), \quad (14)$$

where the terms f^0 and \bar{f}^0 are additionally subject to the relations

$$(f^0, e_{(i)})_{\omega(z)} = 0, \quad i = 1, 2, \quad \bar{f}_3^0(z) = 0, \quad \bar{f}^0(z) = \bar{f}_1^0(z)e_{(1)} + \bar{f}_2^0(z)e_{(2)}, \quad (15)$$

$(\cdot, \cdot)_\Xi$ is the natural inner product in the scalar or vector-valued Lebesgue space $L^2(\Xi)$, and

$$\omega(z) = \{y \in \mathbb{R}^2 : H(z)^{-1}y \in \omega\}. \quad (16)$$

Due to the linearity of the problems (7) and (9), we can make the volume forces f to satisfy the requirements (14) and (15) by multiplying them with normalizing factors and adding zero terms, if necessary. In this section, we only perform the formal asymptotic analysis, so that f^0 and \bar{f}^0 are assumed smooth and the small remainder \tilde{f} is ignored, compare (81).

The asymptotic ansatz for the solution of the problem (7)-(9) is of the usual form

$$u \sim h^{-2}U^{-2}(z) + h^{-1}U^{-1}(\eta, z) + h^0U^0(\eta, z) + h^1U^1(\eta, z) + \dots \quad (17)$$

compare Sanchez-Hubert and Sanchez-Palencia,⁶ Nazarov,⁸ Panasenko,⁹ and others. The first two terms in (17) are independent of the material properties and geometric forms, namely,

$$\begin{aligned} U^{-2}(\eta, z) &= e_{(1)}w_1(z) + e_{(2)}w_2(z), \\ U^{-1}(\eta, z) &= e_{(3)}w_3 + 2^{-1/2}(\eta_1e_{(2)} - \eta_2e_{(1)})w_4 - e_{(3)}\sum_{j=1}^2\eta_j\frac{\partial w_j}{\partial z}(z). \end{aligned} \quad (18)$$

We insert (17), (11), and (12) into Equation (7). Collecting coefficients of the same powers of h yields a recursive family of problems in the planar domain (16). These problems depend on the parameter $z \in Y := (-1, 1)$ and read as

$$\begin{aligned} L^0U^k &= F^k := -L^1U^{k-1} - L^2U^{k-2} + \delta_{k,1}f^0 \quad \text{in} \quad \omega(z), \\ B^0U^k &= G^k := -B^1U^{k-1} \quad \text{on} \quad \partial\omega(z), \quad k = -2, -1, 0, 1. \end{aligned} \quad (19)$$

The following statement is known (see, for instance, Fichera³¹ by $H^{1/2}(\partial\omega(z))$, we denote the Sobolev-Slobodetskii space).

Lemma 2.1. *The problem*

$$L^0 U = F \quad \text{in } \omega(z), \quad B^0 U = G \quad \text{on } \partial\omega(z) \quad (20)$$

with the right-hand sides $F \in L^2(\omega(z))^3$, $G \in H^{1/2}(\partial\omega(z))^3$ has a solution $U \in H^2(\omega(z))^3$ if and only if the compatibility conditions

$$\int_{\omega(z)} d(\eta) F(\eta, z) d\eta + \int_{\partial\omega(z)} d(\eta) G(\eta, z) ds_\eta = 0 \in \mathbb{R}^4, \quad d(\eta)^\top = \begin{pmatrix} 1 & 0 & 0 & -\sqrt{2}\eta_2/2 \\ 0 & 1 & 0 & \sqrt{2}\eta_1/2 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad (21)$$

are satisfied. The solution $U \in H^2(\omega(z))$ is defined up to an additive term in the linear space $\mathbf{L} = \{d(\eta)^\top \mathbf{c} : \mathbf{c} \in \mathbb{R}^4\}$. The solution becomes unique provided that the orthogonality conditions

$$\int_{\omega(z)} d(\eta) U(\eta, z) d\eta = 0 \in \mathbb{R}^4 \quad (22)$$

are imposed, and it gets the same smoothness in $z \in Y$ as the right-hand sides F and G .

Remark 2.2. According to (16) and (2), the family of problems (20) depends on parameter z , while the domain $\omega(z)$ vanishes in the limit $z \rightarrow \pm 1$. Hence, appropriate estimates of solutions must involve parameter-dependent Sobolev norms.³² For example, in the case $G = 0$, the coefficient C in the estimate

$$\|\nabla_\eta^2 U(\cdot, z); L^2(\omega(z))\| + (1 - z^2)^{-2} \|U(\cdot, z); L^2(\omega(z))\| \leq C \|F(\cdot, z); L^2(\omega(z))\|$$

of the solution to (20) and (22) is independent of $z \in Y$ and F . We however will not need to apply results of this kind.

One verifies directly that (13) leads to

$$D_\eta U^{-1} + D_z U^{-2} = 0. \quad (23)$$

The equality (23) implies the validity of (19) with $k = -1$. By (23) and (19), we have

$$\begin{aligned} F^0 &= D_\eta^\top A D_z U^{-1} + D_z^\top A (D_\eta U^{-1} + D_z U^{-2}) = D_\eta^\top A D_z U^{-1}, \\ G^0 &= -D_v^\top A D_z U^{-1} - D_z^\top A (D_\eta U^{-1} + D_z U^{-2}) = -D_v^\top A D_z U^{-1}. \end{aligned} \quad (24)$$

Thus, we can write

$$D_z U^{-1} = \mathcal{Y}(\eta) D(\partial_z) w(z), \quad (25)$$

$$\mathcal{Y}(\eta) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -\eta_1 \\ 0 & 0 & 0 & 0 & 0 & -\eta_1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -\eta_2/2 & \eta_1/2 & 0 \end{pmatrix}^\top, \quad D(\partial_z) = \begin{pmatrix} \partial_z^2 & 0 & 0 & 0 \\ 0 & \partial_z^2 & 0 & 0 \\ 0 & 0 & \partial_z & 0 \\ 0 & 0 & 0 & \partial_z \end{pmatrix}^\top. \quad (26)$$

According to (25) and (24), the solution U^0 of the problem (19) with $k = 0$ takes the form

$$U^0(\eta, z) = \mathcal{X}(\eta, z) D(\partial_z) w(z), \quad (27)$$

where the 3×4 matrix $\mathcal{X} = (\mathcal{X}^1, \dots, \mathcal{X}^4)$ solves the problem

$$-D_\eta^\top A D_\eta \mathcal{X} = F := D_\eta^\top A \mathcal{Y} \quad \text{in } \omega(z), \quad D_v^\top A D_\eta \mathcal{X} = G := -D_v^\top A \mathcal{Y} \quad \text{on } \partial\omega(z). \quad (28)$$

The compatibility conditions (21) follow by integration by parts, in view of the formula $D_{\eta d}(\eta) = 0$.

2.3 | Derivation of the model

For $k = 1$, the right-hand sides in (19) read as

$$F^1 = D_\eta^{\top A} D_z U^0 + D_z^{\top A} (D_{\eta U}^0 + D_z U^{-1}) + f^0, G^1 = -D_v^{\top A} D_z U^0 - D_0^{\top A} (D_{\eta U}^0 + D_z U^{-1}).$$

Let Y be a smooth scalar-valued function on Ω_1 (see (1)). The equality

$$\frac{d}{dz} \int_{\omega(z)} Y(z, \eta) d\eta = \int_{\omega(z)} \frac{\partial Y}{\partial z}(z, \eta) d\eta - \int_{\partial\omega(z)} Y(z, \eta) \partial_z H(z) ds_\eta \quad (29)$$

holds for $z \in \bar{Y}$ and can be easily verified. For any \mathbb{R}^3 -valued smooth function \mathcal{W} , we obtain from (29) that

$$\int_{\omega(z)} D_z^{\top} \mathcal{W}(\eta, z) d\eta - \int_{\partial\omega(z)} D_0^{\top} \mathcal{W}(\eta, z) ds_\eta = D_z^{\top} \int_{\omega(z)} \mathcal{W}(\eta, z) d\eta. \quad (30)$$

Choosing $\mathcal{W} = A(D_\eta U^0 + D_z U^{-1})$ in (30) yields

$$\begin{aligned} I_j &= \int_{\omega(z)} e_{(j)}^{\top} F^1 d\eta + \int_{\partial\omega(z)} e_{(j)}^{\top} G^1 ds_\eta, \\ &= \int_{\omega(z)} f_j^0 d\eta + e_{(j)}^{\top} \left(\int_{\omega(z)} D_\eta^{\top} A D_z U^0 d\eta - \int_{\partial\omega(z)} D_v^{\top} A D_z U^0 ds_\eta + \int_{\omega(z)} D_z^{\top} A (D_\eta U^0 \right. \\ &\quad \left. + D_z U^{-1}) d\eta - \int_{\partial\omega(z)} D_0^{\top} A (D_\eta U^0 + D_z U^{-1}) ds_\eta \right), \\ &= \int_{\omega(z)} (D_z e_{(j)})^{\top} A (D_\eta U^0 + D_z U^{-1}) d\eta + \int_{\omega(z)} f_j^0 d\eta \quad j = 1, 2, 3. \end{aligned} \quad (31)$$

The equalities $I_1 = I_2 = 0$, that is, the first and second components of the compatibility condition (21) are derived using (14). In fact, taking (13) into account, we have

$$\begin{aligned} I_i &= \frac{d}{dz} \int_{\omega(z)} (D_3 e_{(i)})^{\top} A (D_\eta U^0 + D_z U^{-1}) d\eta, \\ &= \frac{d}{dz} \int_{\omega(z)} (D_\eta \eta_i)^{\top} A (D_\eta U^0 + D_z U^{-1}) d\eta, \\ &= -e_{(3)}^{\top} \frac{d}{dz} \left(\int_{\omega(z)} \eta_i D_\eta^{\top} A (D_\eta U^0 + D_z U^{-1}) d\eta - \int_{\partial\omega(z)} \eta_i D_v^{\top} A (D_\eta U^0 + D_z U^{-1}) ds_\eta \right) = 0, \quad i = 1, 2. \end{aligned}$$

By (25) and (27), the third component of the compatibility condition (31) is equivalent to

$$-\frac{d}{dz} \int_{\omega(z)} (D_3 e_{(3)})^{\top} A (D_\eta \mathcal{X}(\eta, z) + \mathcal{Y}(\eta)) d\eta D(\partial_z) w(z) = \mathbf{f}_3(z), \quad (32)$$

$$\mathbf{f}_3(z) := (f^0(z, \cdot), e_{(3)})_{\omega(z)}. \quad (33)$$

We consider the fourth component of the compatibility condition (31) by using the vector $\theta = 2^{-1/2}(\eta_1 e_{(2)} - \eta_2 e_{(1)})$ and write

$$\begin{aligned} 0 &= \int_{\omega(z)} \theta^\top F^1 d\eta + \int_{\partial\omega(z)} \theta^\top G^1 ds_\eta, \\ &= \int_{\omega(z)} \theta^\top f^0 d\eta + \left(\int_{\omega(z)} \theta^\top D_\eta^\top A D_z U^0 d\eta - \int_{\partial\omega(z)} \theta^\top D_v^\top A D_z U^0 ds_\eta \right) \\ &\quad + \left(\int_{\omega(z)} \theta^\top D_z^\top A (D_\eta U^0 + D_z U^{-1}) d\eta - \int_{\partial\omega(z)} \theta^\top D_0^\top A (D_\eta U^0 + D_z U^{-1}) ds_\eta \right). \end{aligned} \quad (34)$$

According to (30), (25), and (27), Equation (34) reduces to

$$-\frac{d}{dz} \int_{\omega(z)} (D_3 \theta(\eta))^\top A(D_\eta \mathcal{X}(\eta, z) + \mathcal{Y}(\eta)) d\eta D(\partial_z) w(z) = \mathbf{f}_4(z), \quad (35)$$

$$\mathbf{f}_4(z) = (f^0(\cdot, z), \theta)_{\omega(z)}. \quad (36)$$

We have now derived two differential equations, (32) and (35), for $w = (w_1, w_2, w_3, w_4)^\top$.

Using (19) with $k = 0, 1$, we get

$$f - L\mathcal{U} = -h^0(L^1 U^1 + L^2 U^0 - \bar{f}^0) - h^1 L^2 U^1 + \bar{f}^0, \quad (37)$$

$$N^{1/2} B\mathcal{U} = -h^1(B^1 U^1 + B^2 U^0) - h^2 B^2 U^1. \quad (38)$$

We next work with coefficients of h^0 in (37) and h^1 in (38) and ignore the other small terms. To make the continuation of the asymptotic procedure possible, we impose the conditions

$$(L^1 U^1 + L^2 U^0 - \bar{f}^0, e_{(i)})_{\omega(z)} + (B^1 U^1 + B^2 U^0, e_{(i)})_{\partial\omega(z)} = 0, \quad i = 1, 2 \quad (39)$$

in the problem (19) for the next term U^2 in (17). We will use (39) to justify the asymptotic procedure in Section 4. By virtue of (30) with $\mathcal{W} = A(D_\eta U^1 + D_z U^0)$, one can rewrite (39) as follows:

$$\partial_z \int_{\omega(z)} (D_3 \eta_i)^\top A(D_\eta U^1 + D_z U^0) d\eta + \bar{f}_i^0 \text{mes}_2 \omega(z) = 0. \quad (40)$$

Identity (13), integration by parts, formula (19) with $k = 1$, and (25), (27) yield

$$\begin{aligned} &\int_{\omega(z)} (D_3 e_{(i)})^\top A(D_\eta U^1 + D_z U^0) d\eta = e_{(3)}^\top \int_{\omega(z)} (D_\eta \eta_i)^\top A(D_\eta U^1 + D_z U^0) d\eta \\ &= e_{(3)}^\top \left(- \int_{\omega(z)} \eta_i D_\eta^\top A(D_\eta U^1 + D_z U^0) d\eta + \int_{\partial\omega(z)} \eta_i D_\eta^\top A(D_\eta U^1 + D_z U^0) ds_\eta \right), \\ &= e_{(3)}^\top \left(\int_{\omega(z)} \eta_i D_\eta^\top A(D_\eta U^0 + D_z U^{-1}) d\eta - \int_{\partial\omega(z)} \eta_i D_\eta^\top A(D_\eta U^0 + D_z U^{-1}) ds_\eta \right) + \int_{\omega(z)} \eta_i f^0 d\eta, \\ &= e_{(3)}^\top \int_{\omega(z)} \eta_i A(D_\eta \mathcal{X} + \mathcal{Y}) d\eta D(\partial_z) w + \int_{\omega(z)} \eta_i f^0 d\eta, \quad i = 1, 2. \end{aligned}$$

Combining this equality with (40) leads to

$$-e_{(3)}^\top \partial_z D_z^\top \int_{\omega(z)} e_{(i)} A(D_\eta \mathcal{X}(\eta, z) + \mathcal{Y}(\eta)) d\eta D(\partial_z) w(z) = \mathbf{f}_i(z), \quad (41)$$

where $i = 1, 2$, and

$$\mathbf{f}_i(z) = \bar{f}_i^0 \text{mes}_2 \omega(z) + \partial_z \int_{\omega(z)} e_{(i)} f_3^0(\eta, z) d\eta. \quad (42)$$

2.4 | The limit system of ordinary differential equations

Note that the following row vectors of length 6, which are related with the left sides of (32), (35), and (41),

$$-\partial_z e_{(3)}^\top D_z^\top \eta_1, -\partial_z e_{(3)}^\top D_z^\top \eta_2, -e_{(3)}^\top D_z^\top, -\theta(\eta)^\top D_z^\top,$$

coincide with rows of the matrix $D(-\partial_z)^\top \mathcal{Y}(\eta)^\top$. Thus, the system (32), (35), and (41) can be written in the condensed matrix form

$$\mathbf{L}(z, \partial_z) w(z) := D(-\partial_z)^\top \mathbf{M}(z) D(\partial_z) w(z) = \mathbf{f}(z), \quad z \in \Upsilon, \quad (43)$$

where $\mathbf{f} = (\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3, \mathbf{f}_4)^\top$ is a column with components (42), (33), (36), and the matrix

$$\begin{aligned} \mathbf{M}(z) &= \int_{\omega(z)} \mathcal{Y}^\top(\eta) A(D_\eta \mathcal{X}(\eta, z) + \mathcal{Y}(\eta)) d\eta \\ &= \int_{\omega(z)} (D_\eta \mathcal{X}(\eta, z) + \mathcal{Y}(\eta))^\top A(D_\eta \mathcal{X}(\eta, z) + \mathcal{Y}(\eta)) d\eta \end{aligned} \quad (44)$$

has size 4. The last equality is obtained by integrating by parts and using relations (28).

The above derivation of the matrix \mathbf{M} is valid for arbitrary anisotropic rods. Let us specify formula (44) for isotropic material. The problem (28) with matrix (6) has the explicit solution (see, e.g., Nazarov⁸, Ch.5)

$$\mathcal{X} = \frac{\lambda}{2(\lambda + \mu)} \begin{pmatrix} (\eta_1^2 - \eta_2^2)/2 & \eta_1 \eta_2 & -\eta_1 & 0 \\ \eta_1 \eta_2 & (\eta_2^2 - \eta_1^2)/2 & -\eta_2 & 0 \\ 0 & 0 & 0 & \sqrt{2}(\lambda + \mu)\lambda^{-1} \varphi(\eta, z) \end{pmatrix}, \quad (45)$$

where φ is the torsion function, that is, a solution of the Neumann problem

$$-\Delta_\eta \varphi(\eta, z) = 0, \quad \eta \in \omega(z), \quad \partial_n \varphi(\eta, z) = \eta_2 n_1(\eta) - \eta_1 n_2(\eta), \quad \eta \in \partial \omega(z)$$

with zero mean value over $\omega(z)$. We put (45) into (44) and obtain that

$$\begin{aligned} \mathbf{M}(z) &= \begin{pmatrix} \mathbf{M}^0(z) & \mathbf{0} \\ \mathbf{0}^\top & \mu G^\omega H(z)^4/2 \end{pmatrix}, \\ \mathbf{M}^0(z) &= \begin{pmatrix} EI_1^\omega H(z)^4 & EI_{12}^\omega H(z)^4 & -E|\omega| p_1^\omega H(z)^3 \\ EI_{12}^\omega H(z)^4 & EI_2^\omega H(z)^4 & -E|\omega| p_2^\omega H(z)^3 \\ -E|\omega| p_1^\omega H(z)^3 & -E|\omega| p_2^\omega H(z)^3 & E|\omega| H(z)^2 \end{pmatrix}, \quad \mathbf{0} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \end{aligned} \quad (46)$$

Here,

$$E = \mu(3\lambda + 2\mu)/(\lambda + \mu), \quad q^\omega(\eta, z) := |\partial_1 \varphi(\eta, z) - \eta_2|^2 + |\partial_2 \varphi(\eta, z) - \eta_1|^2,$$

$p^\omega = (p_1^\omega, p_2^\omega)^\top$ is the mass center of ω , I_{\dots}^ω are the moments of inertia, and G^ω is the torsional stiffness,

$$p_i^\omega = |\omega|^{-1} \int_{\omega} \tau_i d\tau, \quad I_i^\omega = \int_{\omega} \tau_i^2 d\tau, \quad I_{12}^\omega = I_{21}^\omega = \int_{\omega} \tau_1 \tau_2 d\tau, \quad G^\omega = \int_{\omega} q^\omega(\tau) d\tau.$$

Explicit formulae for the integral characteristic G^ω are known for some special cross sections (e.g. see Pólya and Szegő's book³³). If ω is, for example, an ellipse with the semi-axes a and b , we have $G^\omega = \pi a^3 b^3 (a^2 + b^2)^{-1}$.

We emphasize that for a general matrix $\mathbf{M}(z)$, (44), it is straightforward to conclude that the orders of the magnitude of its entries as $z \rightarrow \pm 1$ are the same as in the isotropic case (46). To see this, one takes into account the dependence of the solution $\mathcal{X}(\eta, z)$ on z and the right-hand sides of the problem (28) involving the matrix (26); recall that $H(z) = O(\rho(z)^m)$ as $z \rightarrow \pm 1$, where $\rho(z) = 1 - |z|$. This behavior is specified in the next lemma.

Lemma 2.3. *Every $w = (w_1, w_2, w_3, w_4)^\top \in C_c^\infty(Y)^4$ satisfies the inequality*

$$\begin{aligned} & \|\rho^{2m} \partial_z^2 w_1; L^2(Y)\|^2 + \|\rho^{2m} \partial_z^2 w_2; L^2(Y)\|^2 \\ & + \|\rho^m \partial_z w_3; L^2(Y)\|^2 + \|\rho^{2m} \partial_z w_4; L^2(Y)\|^2 \leq cQ(w, w) \end{aligned} \quad (47)$$

involving the bilinear form $Q(w, v) := (\mathbf{M}(z)D(\partial_z)w(z), D(\partial_z)v)_{\omega(z)}$.

Proof. Let $\xi = (\xi_1, \xi_2, \xi_3, \xi_4)^\top \in \mathbb{R}^4$ and set

$$\mathcal{Z}(\eta, z) = D_\eta \mathcal{X}(\eta, z) + \mathcal{Y}(\eta), \quad \mathbf{Z}(\eta, z) = \mathcal{Z}(\eta, z)H(z)^{-1}, \quad (48)$$

$$\zeta(z) = \xi H(z) \in \mathbb{R}^4, \quad H(z) = \text{diag}\{H(z)^2, H(z)^2, H(z)^1, H(z)^2\}, \quad z \in Y. \quad (49)$$

By the definition (44) of the matrix \mathbf{M} , we have

$$\xi^\top \mathbf{M}(z) \xi = \zeta(z)^\top \int_{\omega(z)} H(z)^{-1} \mathcal{Z}(\eta, z)^\top A \mathcal{Z}(\eta, z) H(z)^{-1} d\eta \zeta(z) = \zeta(z)^\top \mathcal{M} \zeta(z), \quad z \in Y,$$

where $\mathcal{M}(z)$ is a symmetric 4×4 matrix, which is positive definite uniformly in $z \in \bar{Y}$,

$$\mathcal{M}(z) = \int_{\omega(z)} \mathbf{Z}(\eta, z)^\top A \mathbf{Z}(\eta, z) d\eta. \quad (50)$$

These properties follow from the same properties of the constant matrix A (see Nazarov and Slutski⁷, Proposition 2.6) and the common order $\rho(z)^{-2}$ of the entries of \mathbf{Z} in (50) (cf. (49) and (46)). Thus,

$$\xi^\top \mathbf{M}(z) \xi \geq c |\zeta(z)|^2 \quad (51)$$

with a constant $c > 0$. The inequality (51) proves (47). \square

3 | STUDYING THE LIMIT PROBLEMS

3.1 | Variational formulation of the problem and function spaces

We introduce the Hilbert spaces $\mathfrak{H}_{1,2}$, \mathfrak{H}_3 , and \mathfrak{H}_4 as the completion of $C_c^\infty(Y)$ (infinitely differentiable and compactly supported functions) with respect to the norms

$$\|w; \mathfrak{H}_{1,2}\| := (\|\rho^{2m} \partial_z^2 w; L^2(Y)\|^2 + \|w; L^2(v)\|^2)^{1/2}, \quad (52)$$

$$\|w; \mathfrak{H}_3\| := (\|\rho^m \partial_z w; L^2(Y)\|^2 + \|w; L^2(v)\|^2)^{1/2}, \quad (53)$$

$$\|w; \mathfrak{H}_4\| := (\|\rho^{2m} \partial_z w; L^2(Y)\|^2 + \|w; L^2(v)\|^2)^{1/2}, \quad (54)$$

respectively, where $v := (-1/2, 1/2) \subset Y$. We associate the system (43) with the variational problem

$$\mathcal{Q}(w, v) := (\mathbf{M}(z)D(\partial_z)w(z), D(\partial_z)v)_{\omega(z)} = F(v) \quad \forall v \in \mathfrak{H} = \mathfrak{H}_{1,2}^2 \times \mathfrak{H}_3 \times \mathfrak{H}_4. \quad (55)$$

As usual, compare Mikhlin,¹¹ the integral identity (55) is obtained by a scalar multiplication of (43) with a test vector function $v \in C_c^\infty(Y)^4$ and an integration by parts. Then, in view of Lemma 2.3, a completion argument allows us to take test functions from the space \mathfrak{H} , since the right-hand side of (55) must be a continuous functional $F \in \mathfrak{H}^*$ in \mathfrak{H} , for example,

$$F(v) = (\mathbf{f}, v)_Y, \quad \mathbf{f} \in L^2(Y)^4.$$

Investigating the solvability of the problem (55) in \mathfrak{H} and the general properties of its solutions is our next objective.

Given $m \in (0, 1)$, we understand by $R_t : Y \rightarrow (0, +\infty)$ a weight function which is equal to 1, if $m \neq t$, and to $|\log \rho(z)/2|^{-1}$, if $m = t$.

Lemma 3.1. *Let $w = (w_1, \dots, w_4)^\top \in \mathfrak{H}$. The norms (52), (53), (54) are equivalent, respectively, with the norms*

$$|||w|||_{1,2} := (\|\rho^{2m} \partial_z^2 w; L^2(Y)\|^2 + \|\rho^{2m-1} R_{3/4} \partial_z w; L^2(Y)\|^2 + \|\rho^{2m-2} R_{1/4} R_{3/4} w; L^2(Y)\|^2)^{1/2}, \quad (56)$$

$$|||w|||_3 := (\|\rho^m \partial_z w; L^2(Y)\|^2 + \|\rho^{m-1} R_{1/2} w; L^2(Y)\|^2)^{1/2}, \quad (57)$$

$$|||w|||_4 := (\|\rho^{2m} \partial_z w; L^2(Y)\|^2 + \|\rho^{2m-1} R_{1/4} w; L^2(Y)\|^2)^{1/2}. \quad (58)$$

Proof. It suffices to deal with $w \in C_c^\infty(Y)^4$. The equivalences of above-mentioned norms are based on variants of the one-dimensional Hardy inequality. For noncritical exponents m , we use the standard Hardy inequality

$$\int_0^l t^{2\beta-1} |W(t)|^2 dt \leq \frac{1}{\beta^2} \int_0^l t^{2\beta+1} |\partial_t W(t)|^2 dt, \quad (59)$$

where $W \in C^1[0, l]$, $\beta < 0$ if $W(0) = 0$ and $\beta > 0$ if $W(l) = 0$. Lemma 3.1 is a consequence of the estimates

$$\|\rho^{2m-1} \partial_z w; L^2(Y)\| \leq c \|\rho^{2m} \partial_z^2 w; L^2(Y)\|, \quad (60)$$

$$\|\rho^{2m-2} w; L^2(Y)\| \leq c \|\rho^{2m-1} \partial_z w; L^2(Y)\|, \quad (61)$$

$$\|\rho^{m-1} w; L^2(Y)\| \leq c \|\rho^m \partial_z w; L^2(Y)\|, \quad (62)$$

$$\|\rho^{2m-1} w; L^2(Y)\| \leq c \|\rho^{2m} \partial_z w; L^2(Y)\|, \quad (63)$$

the proofs of which depend on parameter m .

Let first $m \in (0, 1/4)$. The estimates (60)-(63) readily follow from (59) by choosing β to be $2m - 1/2$, $2m - 3/2$, $m - 1/2$, or $2m - 1/2$, respectively. We also obtain (61) for $m \in (1/4, 1/2) \cup (1/2, 3/4)$ and (62) for $m \in (1/4, 1/2)$ directly from (59) with the same β .

To estimate $\|\rho^{2m-1} \partial_z w; L^2(Y)\|$ and $\|\rho^{2m-1} w; L^2(Y)\|$ for $m \in (1/4, 1/2)$, we introduce a smooth cutoff function \mathbf{X} such that $\mathbf{X} = 1$ for $z \in Y \setminus v$ and $\mathbf{X} = 0$ for $|z| < 1/4$. We then apply (59) with $l = 1$ and $t = 1 \pm z$ to obtain that

$$\begin{aligned} & \|\rho^{2m-1} \partial_z w; L^2(Y)\| \\ & \leq c(\|\rho^{2m} \partial_z (\mathbf{X} \partial_z w); L^2(Y)\| + \|\partial_z w; L^2(v)\|) \leq c(\|\rho^{2m} \partial_z^2 w; L^2(Y)\| + \|\partial_z w; L^2(v)\|) \\ & \leq c(\|\rho^{2m} \partial_z^2 w; L^2(Y)\| + \|\partial_z^2 w; L^2(v)\| + \|w; L^2(v)\|) \leq c(\|\rho^{2m} \partial_z^2 w; L^2(Y)\| + \|w; L^2(v)\|). \end{aligned}$$

A similar estimate valid for $\rho^{2m-1}w$. We apply the same argument to prove also (60), (62), (63) for $m \in (1/2, 3/4)$ and (60)-(63) for $m \in (3/4, 1)$.

For the critical exponents $m = 1/2$, $m = 1/4$, and $m = 3/4$, we proceed as above, replacing (59) by the Hardy inequalities with logarithms

$$4 \int_0^l t |\partial_t W(t)|^2 dt \geq \int_0^l t^{-1} \left(\log \frac{2l}{t} \right)^{-2} |W(t)|^2 dt, \quad W \in C^1(0, l), \quad W(0) = 0, \quad (64)$$

$$64 \int_0^l t^{-1} \log \left(\frac{2l}{t} \right)^{-2} |\partial_t W(t)|^2 dt \geq \int_0^l t^{-3} \left(\log \frac{2l}{t} \right)^{-2} |W(t)|^2 dt, \quad W \in C^1(0, l), \quad W(0) = 0, \quad (65)$$

(see, e.g., Campbell et al.¹², Lemma 2.1). In particular, we use (64) for (56) and (57), and (65) for (58). \square

3.2 | Boundary conditions

It is known (see, e.g., Nazarov and Plamenevsky³⁵, Ch. 4, §5) that if $\rho^\beta \partial_z W \in L^2(Y)$ for some $\beta < -1/2$, then the limits of $W(z)$ exist, as $z \rightarrow \pm 1$, and if $\rho^{\beta-1} W \in L^2(Y)$, $\beta < -1/2$, then these limits vanish. Let now $w = (w_1, w_2, w_3, w_4) \in \mathfrak{H}$. According to Lemma 3.1, the functions $w_j \in \mathfrak{H}_{1,2}$, $j = 1, 2$ meet for $0 < m < 1/4$ the boundary conditions

$$w_j(\pm 1) = 0, \quad (66)$$

$$\partial_z w_j(\pm 1) = 0. \quad (67)$$

In the case $1/4 < m < 3/4$ only the boundary condition (66) holds for $w_j \in \mathfrak{H}_{1,2}$, $j = 1, 2$. In the case $0 < m < 1/2$ (respectively, $0 < m < 1/4$), the functions $w_3 \in \mathfrak{H}_3$ (respectively $w_4 \in \mathfrak{H}_4$) satisfy the boundary conditions

$$w_3(\pm 1) = 0, \quad (68)$$

respectively,

$$w_4(\pm 1) = 0. \quad (69)$$

The next lemma implies the failure of the homogeneous boundary conditions for $w \in \mathfrak{H}$ in the case $m \geq 1/4$. We denote by $C_0(\overline{Y})$ the subspace $C(\overline{Y})$ consisting of functions which vanish at $z = \pm 1$.

Lemma 3.2.

- (i) If $m \geq 1/4$, the space $C^\infty(\overline{Y})$ is contained in \mathfrak{H}_4 .
- (ii) If $m \geq 1/2$, the space $C^\infty(\overline{Y})$ is contained in \mathfrak{H}_3 .
- (iii) If $m \geq 3/4$, the space $C^\infty(\overline{Y})$ is contained in $\mathfrak{H}_{1,2}$.
- (iv) If $m \geq 1/4$, the space $w \in C^\infty(\overline{Y}) \cap C_0(\overline{Y})$ is contained in $\mathfrak{H}_{1,2}$.

Proof. We introduce smooth cutoff functions

$$\chi(t) = 1 \quad \text{for } t \leq 1/3 \quad \text{and} \quad \chi(t) = 0 \quad \text{for } t \geq 2/3, \quad (70)$$

$$\mathcal{X}_\delta(z) = 1 - \chi(\delta^{-1}(1 - z)) - \chi(\delta^{-1}(1 + z)), \quad z \in Y, \quad (71)$$

where $\delta > 0$ is a small parameter. Clearly,

$$\mathcal{X}_\delta(z) = 1 \quad \text{for } |z| < 1 - 3\delta/2, \quad \mathcal{X}_\delta(z) = 0 \quad \text{for } |z| > 1 - \delta/3, \quad |\partial_z^k \mathcal{X}_\delta(z)| \leq c_k \delta^{-k}, \quad k = 1, 2, \dots$$

Given a function $w \in C(\bar{Y})$, we have the inequalities

$$\begin{aligned}
 \|\mathcal{X}_\delta w; \mathfrak{H}_4\|^2 &\leq C \int_Y (\rho(z)^{4m} |\partial_z \mathcal{X}_\delta(z)|^2 |w(z)|^2 + \mathcal{X}_\delta(z)^2 \rho(z)^{4m} |\partial_z w(z)|^2) d\rho \\
 &\leq c^{(2)} \int_0^\delta \rho^{4m} (\delta^{-2} + 1) d\rho \leq C^{(2)} \delta^{4m-1}, \\
 \|\mathcal{X}_\delta w; \mathfrak{H}_3\|^2 &\leq C \int_Y (\rho(z)^{2m} |\partial_z \mathbf{X}_\delta(z)|^2 |w(z)|^2 + \mathbf{X}_\delta(z)^2 \rho(z)^{2m} |\partial_z w(z)|^2) d\rho \\
 &\leq c^{(1)} \int_0^\delta \rho^{2m} (\delta^{-2} + 1) d\rho \leq C^{(1)} \delta^{2m-1}, \\
 \|\mathbf{X}_\delta w; \mathfrak{H}_{1,2}\|^2 &\leq C \int_Y (\rho(z)^{4m} (|\partial_z^2 \mathbf{X}_\delta(z)|^2 |w(z)|^2 + \partial_z \mathbf{X}_\delta(z)^2 |\partial_z w(z)|^2) + \mathbf{X}_\delta^2 \rho^{4m} |\partial_z^2 w(z)|^2) d\rho \\
 &\leq c^{(0)} \int_0^\delta \rho^{4m} (\delta^{-4} + \delta^{-2} + 1) d\rho \leq C^{(0)} \delta^{4m-3}.
 \end{aligned}$$

If in addition $w \in C_0(\bar{Y})$, then the last exponent reduces from $4m - 3$ to $4m - 1$, because the term with δ^{-4} in the last integral is replaced by δ^{-2} . Since the right-hand sides of the above inequalities tend to 0 as $\delta \rightarrow 0$, we find that any function $w \in C^\infty(\bar{Y})$ can be approximated by $(1 - \mathbf{X}_\delta)w \in C_c^\infty(Y)$ in the norms (58), (56), and (57), respectively.

The above calculations complete the proof except in the cases $m = 1/4$ for \mathfrak{H}_4 , $m = 1/2$ for \mathfrak{H}_3 , and $m = 1/4$, $m = 3/4$ for $\mathfrak{H}_{1,2}$, for which we employ another cutoff function $\mathbf{X}_\delta^0(z) = \mathbf{X}_\delta(\log \rho(z)/|\log \delta|)$. Note that $|\partial_z^k \mathbf{X}_\delta^0(z)| \leq C_k |\log \delta|^{-1} \rho(z)^{-k}$, $k = 1, 2$. We obtain

$$\begin{aligned}
 \|\mathbf{X}_\delta^0 w; \mathfrak{H}_4\|^2 &\leq c \left(\int_\delta^{\sqrt{\delta}} \rho |\log \delta|^{-2} \rho^{-2} d\rho + \int_0^{\sqrt{\delta}} \rho d\rho \right) \leq \frac{C}{|\log \delta|} \quad \text{for } m = \frac{1}{4}, \\
 \|\mathbf{X}_\delta^0 w; \mathfrak{H}_3\|^2 &\leq c \left(\int_\delta^{\sqrt{\delta}} \rho |\log \delta|^{-2} \rho^{-2} d\rho + \int_0^{\sqrt{\delta}} \rho d\rho \right) \leq \frac{C}{|\log \delta|} \quad \text{for } m = \frac{3}{4}, \\
 \|\mathbf{X}_\delta^0 w; \mathfrak{H}_{1,2}\|^2 &\leq c \left(\int_\delta^{\sqrt{\delta}} \rho^3 (\rho^{-4} |\log \delta|^{-2} + \rho^{-2} |\log \delta|^{-2}) d\rho + \int_0^{\sqrt{\delta}} \rho^3 d\rho \right) \leq \frac{C}{|\log \delta|} \quad \text{for } m = \frac{3}{4}.
 \end{aligned}$$

For $m = 1/4$, the inclusion $w \in C^\infty(\bar{Y}) \cap C_0(\bar{Y})$ implies $|w(z)| \leq C\rho(z)$ and yields

$$\|\mathbf{X}_\delta^0 w; \mathfrak{H}_{1,2}\|^2 \leq c \left(\int_\delta^{\sqrt{\delta}} \rho^1 \rho^{-2} |\log \delta|^{-2} d\rho + \int_0^{\sqrt{\delta}} \rho d\rho \right) \leq \frac{C}{|\log \delta|},$$

which completes the proof of the case (iv). □

The next assertion follows from Lemmas 3.1 and 3.2.

Theorem 3.3. *A vector function $w \in \mathfrak{H}$ meets the boundary conditions (66)–(69) for $m \in (0, 1/4)$; (66), (68) for $m \in [1/4, 1/2)$; and (66) for $m \in [1/2, 3/4)$. For $m \in [1/4, 1/2)$ (respectively, $m \in [1/2, 3/4)$, $m \in [3/4, 1)$), there exist functions in \mathfrak{H} such that (67), (69) (respectively, (67)–(69), (66)–(69)) fail.*

3.3 | Solvability of the problem

The quadratic form $Q(w, w)$ vanishes on the space

$$\mathcal{L} = \{w(z) = (c_1 + c_1^1 z, c_2 + c_2^1 z, c_3, c_4) \mid c_q^p \in \mathbb{R}\}.$$

By $\mathcal{L}^{(m)}$, we understand the subspace $\mathcal{L} \cap \mathfrak{H}$. Our consideration in Section 97. shows that

$$\begin{aligned} \mathcal{L}^{(m)} &= \{0\} \quad \text{for } m \in (0, 1/4], \\ \mathcal{L}^{(m)} &= \{w(z) = (0, 0, 0, c_4)^\top\} \quad \text{for } m \in (1/4, 1/2], \\ \mathcal{L}^{(m)} &= \{w(z) = (0, 0, c_3, c_4)^\top\} \quad \text{for } m \in (1/2, 3/4], \\ \mathcal{L}^{(m)} &= \{w(z) = (c_1, c_2, c_3, c_4)^\top\} \quad \text{for } m \in (3/4, 1). \end{aligned} \quad (72)$$

Clearly, $Q(w, w)$ is positive definite only in the case $(0, 1/4)$.

Now the Riesz representation theorem yields the following assertion.

Theorem 3.4. *The problem (55) with the right-hand side $F \in \mathfrak{H}^*$ has a solution $w \in \mathfrak{H}$ if and only if*

$$F(\ell) = 0 \quad \forall \ell \in \mathcal{L}^{(m)}.$$

This solution is defined up to a summand in $\mathcal{L}^{(m)}$, but under the orthogonality conditions

$$(w, \ell)_Y = 0 \quad \forall \ell \in \mathcal{L}^{(m)},$$

it becomes unique and meets the estimate

$$\|w; \mathfrak{H}\| \leq c \|F; \mathfrak{H}^*\|.$$

Notice that the problem (55) has a unique solution for all data only in the case $m \in (0, 1/4)$.

3.4 | Boundary layer phenomenon

The fact that the limit problem (55) can have no solution for some \mathbf{f} , if $m \geq 1/4$, means that the system (43) with boundary conditions mentioned in Theorem 3.3 can not serve as a limit problem for $m \geq 1/4$. A proper limit problem can be found by involving a more complicated ansatz in the asymptotic analysis. To this end, we consider a neighborhood of the point $(0, 0, -1)$, make the coordinate dilation,

$$x \mapsto \xi = (\delta^{-1}x_1, \delta^{-1}x_2, \delta^{-1}(x_3 + 1)) \quad \text{with } \delta = h^{1/(1-m)} \quad (73)$$

and set $h = 0$ formally. According to (2), these transform the domain (1) into the paraboloid

$$\Pi = \{\xi : H^{-m} \xi_3^{-m}(\xi_1, \xi_2) \in \omega\}, \quad (74)$$

and a similar paraboloid is made from the other end of the rod. The construction of the boundary layer requires the following mixed boundary value problem of elasticity theory in Π :

$$L(\nabla_\xi)W(\xi) = F(\xi), \quad \xi \in \Pi, \quad W(\xi) = 0, \quad \xi \in \Gamma, \quad B(\nabla_\xi)W(\xi) = 0, \quad \xi \in \partial\Pi \setminus \Gamma, \quad (75)$$

where Γ is a non-empty compact subset of the surface $\partial\Pi$.

Remark 3.5. The change (73) and the Dirichlet condition in (75) explain our choice of the radius

$$r(h) = r_0 h^{1/(1-m)}, \quad r_0 > 0, \quad (76)$$

in the definition of the terminals Γ_h^\pm ; see (8).

As usual, the variational problem

$$(AD(\nabla_\xi)W, D(\nabla_\xi)V)_\Pi = \mathfrak{F}(V) \quad \forall V \in \mathfrak{G}, \quad (77)$$

which serves for (75) is posed in the energy space \mathfrak{E} defined as the completion of $C_c^\infty(\bar{\Pi} \setminus \Gamma)^3$ in the energy norm $\|D(\nabla_\xi)W; L^2(\Pi)\|$. Clearly, due to the Dirichlet condition on Γ , the problem (77) has a unique energy solution $W \in \mathfrak{E}$ for any functional $\mathfrak{F} \in \mathfrak{E}^*$.

However, we now refer to Nazarov¹⁴ for the discovery of certain peculiar features of elastic paraboloids, in particular, the necessity to distinguish between the above-mentioned “energy solution” and “the solution with finite energy.” For example, in the case $m \in (0, 1/4)$, the homogeneous problem (75) (with $F = 0$) has solutions of the form

$$\mathbf{v}(\xi) = \mathbf{r}(\xi) + \mathbf{a}(\xi), \quad (78)$$

where \mathbf{r} is a rigid motion (a linear combination of the three translations and three rotations) and $\mathbf{a} \in \mathfrak{E}$ is the energy component. Since $D(\nabla_\xi)\mathbf{r} = 0$, the displacement field (78) still possesses finite elastic energy, although it is not in the energy space because, according to Nazarov,¹⁴ a rigid motion cannot be approximated by vectors of $C_c^\infty(\bar{\Pi})^3$ in the energy norm, if $m \in (0, 1/4)$. In the case $m \geq 1/4$ the situation is different, and in fact, the dimension of the space $\mathfrak{L}^{(m)}$ of solutions with finite energy (equal to 6 for $m \in (0, 1/4)$) reduces stepwise when the exponent m in (74) grows. As it is easy to predict, the changes of the dimension occur at the critical values $m = 1/4, 1/2, 3/4$. Furthermore, for the dimensions of these spaces and those in (72), there holds the formula

$$\dim \mathfrak{L}^{(m)} + \dim \mathcal{L}^{(m)} = 6.$$

A thorough explanation of these phenomena, the completion of the construction of the boundary layers and the compensation of the absence of the solutions to the limit problem (55) in the case $m \in [1/4, 1)$ will be the subject of a planned forthcoming paper.

4 | JUSTIFICATION OF THE ASYMPTOTICS

4.1 | Preliminaries

In this section, we justify the derived asymptotic ansatz (17) in the case $m \in (0, 1/4)$, where all six boundary conditions (66)–(69) are imposed at the endpoints $z = \pm 1$ and the limit problem has a unique solution.

We recall the following weighted Korn inequality in Nazarov et al.¹³, Theorem 1.1.

$$|||u; \Omega_h|||^2 \leq C\mathcal{E}(u; \Omega_h), \quad \mathcal{E}(u; \Xi) = \frac{1}{4} \sum_{i,j=1}^3 \int_{\Xi} \left(\frac{\partial u_i}{\partial x_j}(x) + \frac{\partial u_j}{\partial x_i}(x) \right)^2 dx, \quad (79)$$

where the displacement vector $u \in H^1(\Omega_h)^3$ is subject to the Dirichlet condition (8) and C is a constant independent of the small parameter h and u . The norm $||| \cdot; \Omega_h |||$ on the left-hand side of (79) is the anisotropic weighted Sobolev norm

$$\begin{aligned} |||u; \Omega_h||| := & \left(\int_{\Omega_h} \left\{ \sum_{\alpha=1}^2 \left(\left| \frac{\partial u_\alpha}{\partial y_\alpha} \right|^2 + h^2 \rho_h(z)^{2m-2} \left(\left| \frac{\partial u_\alpha}{\partial z} \right|^2 + \left| \frac{\partial u_3}{\partial y_\alpha} \right|^2 \right) + h^2 \rho_h(z)^{2m-4} |u_\alpha|^2 \right) \right. \right. \\ & \left. \left. + h^2 \rho_h(z)^{2m-2} \left(\left| \frac{\partial u_1}{\partial y_2} \right|^2 + \left| \frac{\partial u_2}{\partial y_1} \right|^2 \right) + \left| \frac{\partial u_3}{\partial z} \right|^2 + \frac{1}{\rho_h(z)^2} |u_3|^2 \right\} dy dz \right)^{1/2} \end{aligned} \quad (80)$$

with $\rho_h(z) = h^{1/(1-m)} + 1 - |z|$; see (1.7) in Nazarov et al.¹³

We fix the radius (76) as in formula (8) and assume that the terms in the representation (14) of the right-hand sides of (7) satisfy the relations (15) and, moreover,

$$\hat{f}^0, \partial_z \hat{f}^0 \in L^2(\omega \times \Upsilon)^3, \quad \bar{f}^0 \in L^2(\Upsilon)^2, \quad \tilde{f} \in L^2(\Omega_h)^3, \quad (81)$$

where $\hat{f}^0(\hat{\eta}, z) = f^0(H(z)\hat{\eta}, z)$. We also set

$$\begin{aligned} \mathcal{N} &= \|\hat{f}^0; L^2(\omega \times \Upsilon)\| + \|\partial_z \bar{f}^0; L^2(\omega \times \Upsilon)\| + \|\bar{f}^0; L^2(\Upsilon)\|, \\ \widetilde{\mathcal{N}} &= h^{\frac{4m-1}{2(1-m)}} \left(h^{-1} \|\rho_h^{2-m} \tilde{f}_1; L^2(\Omega_h)\| + h^{-1} \|\rho_h^{2-m} \tilde{f}_2; L^2(\Omega_h)\| + \|\rho_h \tilde{f}_3; L^2(\Omega_h)\| \right). \end{aligned} \quad (82)$$

We will prove the estimate

$$|||u - \mathbf{u}; \Omega_h||| \leq c\mathcal{E}(u - \mathbf{u}; \Omega_h) \leq Ch^{\frac{1-4m}{2(1-m)}}(\mathcal{N} + \widetilde{\mathcal{N}})$$

for the difference between the true u and approximate \mathbf{u} solutions. Since $m \in (0, 1/4)$, the factor $h^{\frac{1-4m}{2(1-m)}}$ is small, and as for the two quantities (82), the first one involves proper norms of functions composing the right-hand side \mathbf{f} of the limit system (43), while the second one establishes the smallness of the remainder \tilde{f} ignored in the formal dimension reduction procedure.

4.2 | Behavior of $w(\mathbf{z})$ as $\mathbf{z} \rightarrow \pm 1$

Our system of ordinary differential equations (43) with degenerate coefficients could be treated with many methods, but we choose to use the efficient Kondratiev theory³⁴ (see also, e.g., Nazarov and Plamenevsky³⁵, Ch. 3 and 6), since it does not lead to any complicated calculations; we only need to introduce some definitions. We thus denote the Kondratiev weighted space $V_\beta^l(\Upsilon)$ as the subspace of $H_{loc}^l(\Upsilon)$ consisting of functions v with finite norm

$$\|v; V_\beta^l(\Upsilon)\| = \left(\sum_{k=0}^l \|\rho^{\beta-l+k} \partial_z^k v; L^2(\Upsilon)\| \right)^{1/2}.$$

We will apply the classical theorem on asymptotics³⁴ (cf. also Nazarov and Plamenevsky³⁵, Thm. 3.5.6, 4.2.1) by regarding the semi-axis as a one-dimensional cone and consequently treating the endpoints of the interval Y as its “corner” points. We emphasize that it is possible for us to apply the Kondratiev theory intrinsically, since Lemma 3.1 implies for $m \neq 1/4, 1/2, 3/4$, the equalities

$$\mathfrak{H}_{1,2} = V_{2m}^2(\Upsilon), \quad \mathfrak{H}_3 = V_m^1(\Upsilon), \quad \mathfrak{H}_4 = V_{2m}^1(\Upsilon).$$

If $m = 1/4, 1/2, 3/4$, the norms (56), (57), and (58) contain additional logarithmic factors, however, the Kondratiev theory still works because we can start with a slightly smaller weight exponent β . For example, in the case of the norms (53) and (57) with $m = 1/2$, we can use the inclusion

$$\mathfrak{H}_3 \subset V_{m+\delta}^1(\Upsilon) \quad \text{for any } \delta > 0.$$

Taking into account the integration area in formulas (42) and (33), (36), the assumption (81) yields

$$\mathbf{f}_1, \mathbf{f}_2 \in V_{1-2m}^0(\Upsilon), \quad \mathbf{f}_3 \in V_{1-m}^1(\Upsilon), \quad \mathbf{f}_4 \in V_{1-2m}^1(\Upsilon) \quad (83)$$

with norm bounds not exceeding $c\mathcal{N}$.

Recalling the degeneration magnitudes of the entries of the matrix (44) as $\mathbf{z} \rightarrow \pm 1$, we see that the differential operator of the system (43) maps as

$$\begin{aligned} \mathbf{L} : DV_\beta(\Upsilon) &:= V_{\beta+2m}^4(\Upsilon)^2 \times V_{\beta+m}^3(\Upsilon) \times V_{\beta+2m}^3(\Upsilon) \\ &\rightarrow \mathcal{R}V_\beta(\Upsilon) := V_{\beta-2m}^0(\Upsilon)^2 \times V_{\beta-m}^0(\Upsilon) \times V_{\beta-2m}^0(\Upsilon); \end{aligned}$$

hence, in view of (83), we have $\mathbf{f} \in \mathcal{R}V_1(\Upsilon)$ and $\|\mathbf{f}; \mathcal{R}V_1(\Upsilon)\| \leq c\mathcal{N}$.

Proposition 4.1. *Let $m \in (0, 1/4)$. Then the problem (43), (66)–(69) has a unique solution $w \in DV_2(\Upsilon)$, which can be written as the decomposition*

$$w(\mathbf{z}) = w^{as}(\mathbf{z}) + \tilde{w}(\mathbf{z}) \quad (84)$$

with the estimate

$$\begin{aligned} \|\tilde{w}; DV_1(\Upsilon)\| + \sum_{j=1,2} \left(\sum_{l=0}^4 \sup_{\mathbf{z} \in Y} \rho(\mathbf{z})^{l-2+4m} |\partial_z^l w_j(\mathbf{z})| + \sum_{l=0}^3 \sup_{\mathbf{z} \in Y} \rho(\mathbf{z})^{l-1+2jm} |\partial_z^l w_{2+j}(\mathbf{z})| \right) \\ \leq c\|\mathbf{f}; \mathcal{R}V_1(\Upsilon)\| \leq C\mathcal{N}. \end{aligned} \quad (85)$$

Although Proposition 3.1 is a direct consequence of the general theory³⁴ (see also Nazarov and Plamenevsky³⁵, Ch.3,5), its proof needs to be commented.

First of all, the unique solution $w \in \mathfrak{H}(\Upsilon)$ of the variational problem (55) with the right-hand side

$$F(v) = (\mathbf{f}, v)_Y, \quad \|F; \mathfrak{H}(\Upsilon)^*\| \leq c\|\mathbf{f}; \mathcal{R}V_2(\Upsilon)\| \leq C\mathcal{N}$$

becomes the classical solution $w \in DV_2(\Upsilon)$ because

$$\begin{aligned}
\|w; DV_2(\Upsilon)\|^2 &\leq c \left(\sum_{j=1,2} \left(\|\partial_z^4 w_j; V_{2+2m}^0(\Upsilon)\|^2 + \|\partial_z^3 w_{2+j}; V_{2+2m_j}^0(\Upsilon)\|^2 \right. \right. \\
&\quad \left. \left. + \sum_{j=1,2} \left(\|w_j; V_{2m-2}^0(\Upsilon)\|^2 + \|w_{2+j}; V_{mj-2}^0(\Upsilon)\|^2 \right) \right) \right) \\
&\leq c \left(\|f; \mathcal{R}V_2(\Upsilon)\|^2 + \|w; \mathfrak{H}(\Upsilon)\|^2 \right).
\end{aligned}$$

Here, we have used Lemma 3.1 with $m \in (0, 1/4)$ and the simple interpolation inequality

$$\|\mathbf{w}; V_\beta^l(\Upsilon)\| \leq c \left(\|\partial_z^l \mathbf{w}; V_\beta^0(\Upsilon)\| + \|\mathbf{w}; V_{\beta-l}^0(\Upsilon)\| \right).$$

To form the decomposition (84)-(85), we need to consider the model problem in the one-dimensional “cone” \mathbb{R}_+ . Following Kondrat’s, ³⁴ we introduce the principal parts of the operator $\mathbf{L}(z, \partial_z)$ at the points $z = \pm 1$ as

$$\mathbf{L}^\pm(\rho, \partial_\rho) = D(\pm \partial_\rho)^\top \mathcal{H}^\pm(\rho) \mathbf{M}^\pm \mathcal{H}^\pm(\rho) D(\mp \partial_\rho), \quad (86)$$

where, according to (2), (49) and (51),

$$\mathcal{H}^\pm(\rho) = \text{diag}\{H_\pm^2 \rho^{2m}, H_\pm^2 \rho^{2m}, H_\pm \rho^m, H_\pm^2 \rho^{2m}\}, \quad (87)$$

$$\mathbf{M}^\pm = \mathcal{H}(z)^{-1} \mathbf{M}(z) \mathcal{H}(z)^{-1} \Big|_{z=\pm 1}. \quad (88)$$

Note that, by Lemma 2.3, the matrices \mathbf{M}^\pm are symmetric and positive. In the model system

$$\mathbf{L}^\pm(\rho, \partial_\rho) \mathbf{w}(\rho) = 0, \rho \in \mathbb{R}_+, \quad (89)$$

we can write the matrix operator (86) as

$$D(\rho^{-1}) \mathcal{H}^\pm(\rho) \mathfrak{L}^\pm(\rho \partial_\rho) \mathcal{H}^\pm(\rho) D(\rho^{-1}),$$

where $\mathfrak{L}^\pm(\lambda)$ is a matrix of polynomials of λ . This leads us to seek for power-law solutions of (89),

$$\mathbf{m}(\rho) = \rho^\lambda \mathcal{H}^\pm(\rho)^{-1} D(\rho) \mathbf{m}^0, \quad (90)$$

compare Nazarov and Plamenevsky.³⁵ Proposition 3.5.4 Inserting (90) into (89), we conclude that the number $\lambda \in \mathbb{C}$ and the column \mathbf{m}^0 must satisfy the algebraic system

$$\mathfrak{L}^\pm(\lambda) \mathbf{m}^0 = 0. \quad (91)$$

Some of the solutions (91) are known a priori, for example, the linear vector functions belonging to \mathcal{L} . In view of (87), these get following eigenvalues λ listed according to multiplicity:

$$2m-2, 2m-2, 2m-1, 2m-1, m-1, 2m-1. \quad (92)$$

The corresponding eigenvectors could be written explicitly, too.

Nor these eigenvectors neither the eigenvalues depend on the matrix (88). The algebraic system (91) has exactly 12 roots. Since the original matrix operator (86) is formally self-adjoint, we can use Nazarov and Plamenevsky³⁵, Lem. 3.5.9, Prop. 3.5.2 to find and list all other eigenvalues,

$$1-2m, 1-2m, -2m, -2m, -m, -2m, \quad (93)$$

which, like the eigenvalues (92), are independent of (88). The eigenvectors of \mathcal{L}^\pm corresponding to (93) however depend on the matrix (88), but we do not need them later.

Remark 4.2.

1. The above-mentioned independence of (88) makes it possible to compute the eigenvalues (93) by using the isotropy matrix (46).
2. Denoting the numbers in (92) by λ_p^+ and those in (93) by λ_p^- , $p = 1, \dots, 6$, they satisfy the identity $1 + \lambda_p^+ + \lambda_p^- = 0$ (cf. Nazarov and Plamenevsky³⁵, Ch.3, §5).
3. Formulas (92) and (93) remain valid for any $m \in (0, 1)$ with exception of the critical values $m = 1/4, 1/2, 3/4$, for which power-law solutions may become linear in $\log \rho$.

The power-law solutions with the exponents (93) belong to the space $DV_2(Y^\pm)$ and describe the asymptotic behavior of the solution of the problem (43), (66)-(69). The linear vector functions generated by the exponents (92) are excluded by the boundary conditions. The smallest number $-2m$ in the list (93) determines the decay properties of the detached term as was demonstrated by the sup norms in (85). Finally, we have

$$w^{as}(z) = \sum_{\pm} \chi_{\pm}(z) \sum_{p=3}^6 a_p^{\pm} \mathfrak{w}^p(1 \mp z), \quad (94)$$

where \mathfrak{w}^p are power-law solutions which correspond to the last four exponents in (93) and therefore do not belong to $DV_2(Y)$. Furthermore, the coefficients a_p^{\pm} meet the bound $|a_p^{\pm}| \leq c\mathcal{N}$ and the cutoff functions are defined as $\chi_{\pm}(z) = \chi(1 \mp z)$, where χ is as in (70). The change of variables $\rho \mapsto t = \log \rho$, the embedding $H^1(\mathbb{R}) \subset C(\mathbb{R})$, and the inclusion $\tilde{w} \in DV_1(Y)$ imply the pointwise estimates

$$\sum_{j=1,2} \left(\sum_{l=0}^3 \sup_{z \in Y} \rho(z)^{l+2m-5/2} |\partial_z^l \tilde{w}_j(z)| + \sum_{l=0}^2 \sup_{z \in Y} \rho(z)^{l+jm-3/2} |\partial_z^l \tilde{w}_{j+2}(z)| \right) \leq c \|\tilde{w}; DV_1(Y)\|. \quad (95)$$

Since the exponents of the powers of $\rho(z)$ in (95) are strictly smaller than the corresponding exponents in (85) (recall $m < 1/4$), it suffices in the following estimates to track carefully only the term (94). This will describe the behavior of the solution (84) as $z \rightarrow \pm 1$.

4.3 | The approximate solution

We deal with the sum of four terms of the ansatz (17),

$$\mathbf{u}(h, x) = h^{-2} U^{-2}(z) + h^{-1} U^{-1}(\eta, z) + h^0 U^0(\eta, z) + h^1 U^1(\eta, z)$$

and define an approximate solution of the problem (7)-(9)

$$\mathcal{U}_X(h, x) = X_h(z) \mathbf{u}(h, x). \quad (96)$$

Here, X_h is the cutoff function \mathbf{X}_δ of (71) with the small parameter $\delta = \frac{2}{3}r(h)$. Noting that

$$\text{supp}(1 - X_h), \quad \text{supp}(\partial_k X_h) \subset \overline{\Omega_h^\bullet}, \quad \text{where} \quad \Omega_h^\bullet = \{x \in \Omega_h : |z| > 1 - r(h)\}$$

we see that the product (96) fulfils the Dirichlet condition (8). Furthermore, the singularities of $w \in DV_2(Y)$ do not matter, because of the factor X^h , hence, taking into account the formulas in Section 2, for example, (17) and (27), \mathcal{U}_X falls into the subspace $H_0^1(\Omega_h; \Gamma_h)^3 \subset H^1(\Omega_h)^3$ of functions satisfying (96).

Lemma 4.3. *Let $m \in (0, 1/4)$. Owing to the inequality (85) for the solution (84) of the limit problem (43), (66)-(69), there holds the estimate*

$$|||\mathbf{u} - \mathcal{U}_X; \Omega_h||| + \|D(\nabla_x)(\mathbf{u} - \mathcal{U}_X), L^2(\Omega_h)\| \leq ch^{\frac{1-4m}{2(1-m)}} \mathcal{N}_0, \quad (97)$$

where the constant c is independent of $h \in (0, 1]$ and $w \in DV_2(Y)$.

Proof. According to (18) and (27), we have

$$|||(1 - X_h)\mathbf{u}; \Omega_h|||_h^2 \leq c \sum_{k=0}^2 h^{-2k} (\|\nabla U^{-k}; L^2(\Omega_h^\bullet)\|^2 + h^{-2/(1-m)} \|U; L^2(\Omega_h^\bullet)\|^2). \quad (98)$$

Let us process the terms on the right-hand side of (98). By (85), the estimate

$$|\partial_z^k w_j(z)| \leq c \mathcal{N}_0 h^{\frac{2-4m-k}{1-m}}, \quad k = 0, 1, \quad j = 1, 2,$$

holds on the subdomain Ω_h^\bullet , therefore,

$$\|\partial_z^k w_j; L^2(\Omega_h^\bullet)\|^2 \leq c \mathcal{N}_0 h^{\frac{4-8m-2k}{1-m}} \text{mes}_3(\Omega_h^\bullet) \leq ch^{\frac{7-8m-2k}{1-m}} \mathcal{N}_0.$$

Thus, we obtain

$$h^{-4} |||U^{-2}; \Omega_h^\bullet|||_h^2 \leq ch^{\frac{1-4m}{2(1-m)}} \mathcal{N}_0.$$

Other terms on the right-hand side of (98) are estimated analogously. Since $D(\mathcal{U}_X) = X_h D(\mathbf{u}) + D_z(X_h)\mathbf{u}$, the right-hand side of (98) is a majorant for $\|D(\nabla_x)(\mathbf{u} - \mathcal{U}_X); L^2(\Omega_h)\|$. \square

4.4 | Estimates of the residues and theorem on asymptotics

We set

$$I(\mathbf{R}) := (AD(\nabla_x)\mathbf{R}, D(\nabla_x)\mathbf{R})_{\Omega_h}, \quad (99)$$

where $\mathbf{R} = u - \mathcal{U}_X$ is the difference of the true and approximate solutions and proceeds to evaluate this scalar product. Substituting (96) into the equality (7) yields

$$\begin{aligned} L\mathcal{U}_X = & L(X_h - 1)\mathbf{u} + h^{-4}L^0U^{-2} + h^{-3}(L^0U^{-1} + L^1U^{-2}) + h^{-2}(L^0U^0 + L^1U^{-1} \\ & + L^2U^{-2}) + h^{-1}(L^0U^1 + L^1U^0 + L^2U^{-1}) + h^0(L^2U^0 + L^1U^1) + h^1L^2U^1 =: \mathbf{F}. \end{aligned}$$

Recalling the asymptotic procedure in Section 2 and considering also the boundary condition (9), we obtain

$$\mathbf{F} = L(X_h - 1)\mathbf{u} + h^0(L^1U^1 + L^2U^0) + h^1L^2U^1, \quad \mathbf{G} = B(X_h - 1)\mathbf{u} + h^1B^1U^1,$$

so that $\mathbf{R} = u - \mathcal{U}_X$ satisfies the boundary value problem

$$L\mathbf{R}(h, x) = f(h, x) - \mathbf{F}(h, x), \quad x \in \Omega_h, \quad B\mathbf{R}(h, x) = \mathbf{G}(h, x), \quad x \in \partial\Omega_h.$$

We take the scalar product of this system with \mathbf{R} , integrate by parts, and use the boundary conditions. This yields formula (99), where the continuous functional I of $\mathbf{R} \in H_0^1(\Omega_h; \Gamma_+ \cup \Gamma_-)^3$ is the sum $\tilde{I} + I_X + I_1 + I_{01} + I_0$ with

$$\begin{aligned} \tilde{I}(\mathbf{R}) &= (\tilde{f}, \mathbf{R})_{\Omega_h}, \quad I_X(\mathbf{R}) = (AD(\nabla_x)(1 - X_h)\mathcal{U}, D(\nabla_x)\mathbf{R})_{\Omega_h}, \quad I_1(\mathbf{R}) = -h(AD_zU^1, D_z\mathbf{R})_{\Omega_h}, \\ I_{01}(\mathbf{R}) &= -(D_\eta AD_zU^1, \mathbf{R})_{\Omega_h}, \quad I_0(\mathbf{R}) = (D_z A(D_\eta U^1 + D_z U^0), \mathbf{R})_{\Omega_h} + (\bar{f}, \mathbf{R})_{\Omega_h}. \end{aligned}$$

By the Korn inequality (79) and (94), (85), we have

$$\begin{aligned} |I_{01}(\mathbf{R}) + I_1(\mathbf{R})| &\leq Ch (\|\rho D_z AD_z U^1; L^2(\Omega_h)\| + \|\rho D_y AD_z U^1; L^2(\Omega_h)\|) \|\rho^{-1}\mathbf{R}; L^2(\Omega_h)\| \\ &\leq Ch \mathcal{N}_0 \mathcal{E}(\mathbf{R}; \Omega_h)^{1/2}. \end{aligned}$$

To evaluate $I_{01}(\mathbf{R})$, we recall the known inequality (see Nazarov⁸, Prop. 3.4.13),

$$\begin{aligned} \|\partial_z (\mathbf{R}_j - \langle \mathbf{R}_j \rangle); L^2(\Omega_h)\|^2 &\leq c\mathcal{E}(\mathbf{R}; \Omega_h), \\ \langle \mathbf{R}_j \rangle(z) &= (\text{mes}_2 \omega_h(z))^{-1} \int_{\omega_h(z)} \mathbf{R}_j(y, z) dy, \quad j = 1, 2. \end{aligned} \quad (100)$$

We now multiply the null expression (40) by $\bar{\mathbf{R}}_j$ and integrate by parts to obtain the equality

$$I_{10}(\mathbf{R}) = - \left(A(D_z U^0 + D_\eta U^1), D_z \left(\mathbf{R} - \sum_{j=1}^2 e_{(j)} \langle \mathbf{R}_j \rangle \right) \right)_{\Omega_h}.$$

Hence, applying (94), (85), and (100) yields the estimate $|I_{10}(\mathbf{R})| \leq ch\mathcal{N}_0\mathcal{E}(\mathbf{R}; \Omega_h)^{1/2}$. Finally, Lemma 4.3 gives $|I_X(\mathbf{R})| \leq Ch^{\frac{1-4m}{2(1-m)}}\mathcal{N}_0\mathcal{E}(\mathbf{R}; \Omega_h)^{1/2}$, and the definition (82) implies

$$|\tilde{I}| \leq Ch^{\frac{1-4m}{2(1-m)}}\widetilde{\mathcal{N}}_0|||\mathbf{R}; \Omega_h||| \leq Ch^{\frac{1-4m}{2(1-m)}}\widetilde{\mathcal{N}}_0\mathcal{E}(\mathbf{R}; \Omega_h)^{1/2}. \quad (101)$$

The estimates (99)-(101), together with the Korn inequality (79), (80), lead us to the main theorem on asymptotics. Recall that the leading asymptotic term \mathbf{u} has been constructed in (18), (27).

Theorem 4.4. *Let $0 < m < 1/4$ and assume that the data f satisfies (15), (81). Then, for the solution u of the problem (7), (8), (9) there holds the estimate*

$$|||u - \mathbf{u}; \Omega_h||| \leq c\mathcal{E}(u - \mathbf{u}; \Omega_h)^{1/2} \leq Ch^{\frac{1-4m}{2(1-m)}}(\mathcal{N}_0 + \widetilde{\mathcal{N}}_0), \quad (102)$$

where \mathcal{N}_0 and $\widetilde{\mathcal{N}}_0$ are as in (82) and the constants $c, C > 0$ depend neither on f , nor on $h \in (0, h_0]$.

Formulas (102) and (79), (80) provide weighted estimates of the components of the vector $u - \mathbf{u}$ and their derivatives. It should be mentioned that

$$|||\mathbf{u}; \Omega_h||| \geq c_0\mathcal{N}, \quad \mathcal{E}(\mathbf{u}; \Omega_h) \geq c_1\mathcal{N}, \quad c_p > 0,$$

so that since $\frac{1-4m}{2(1-m)} > 0$ in our case $m < 1/4$, the relation (102) indeed justifies the constructed asymptotics.

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